Groverian Entanglement Measure of Pure Quantum States with Arbitrary Partitions

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The Groverian entanglement measure of pure quantum states of n qubits is generalized to the case in which the qubits are divided into any $m \leq n$ parties and the entanglement between these parties is evaluated. To demonstrate this measure we apply it to general states of three qubits and to symmetric states with any number of qubits such as the Greenberg-Horne-Zeiliner state and the W state.

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I. INTRODUCTION

The potential speedup offered by quantum computers is exemplified by Shor's factoring algorithm [1], Grover's search algorithm [2, 3], and algorithms for quantum simulation [4]. Although the origin of this speed-up is not fully understood, there are indications that quantum entanglement plays a crucial role [5, 6]. In particular, it was shown that quantum algorithms that do not create entanglement can be simulated efficiently on a classical computer [7]. It is therefore of interest to quantify the entanglement produced by quantum algorithms and examine its correlation with their efficiency. This requires to develop entanglement measures for the quantum states of multiple qubits that appear in quantum algorithms.

The special case of bipartite entanglement has been studied extensively in recent years. It was established as a resource for quantum teleportation procedures. The entanglement of pure bipartite states can be evaluated by the von Neumann entropy of the reduced density matrix, traced over one of the parties. For mixed bipartite states, several measures were proposed, namely entanglement of formation and entanglement of distillation [8, 9]. In particular, for states of two qubits an exact formula for the entanglement of formation was obtained [10, 11]. Bipartite pure states of more than two qubits were also studied. It was shown that generic quantum states can be reconstructed from a fraction of the reduced density matrices, obtained by tracing over some of the qubits [12, 13].

The more general case of multipartite entanglement is not as well understood. Recent work based on axiomatic considerations has provided a set of properties that entanglement measures should satisfy [14, 15, 16, 17]. These properties include the requirement that any entanglement measure should vanish for product (or separable) states, it should be invariant under local unitary operations and should not increase as a result of any sequence of local operations complemented by only classical communication between the parties. Quantities that satisfy these properties are called entanglement monotones. These properties, that should be satisfied for bipartite as well as multipartite entanglement, provide useful guidelines in the search for entanglement measures for multipartite quantum states. One class of entanglement measures, based on metric properties of the Hilbert space was proposed and shown to satisfy these requirements [14, 15, 18]. Another class of measures, based on polynomial invariants has been studied in the context of multipartite entanglement [19, 20]. However, the connection between such measures and the efficiency of quantum algorithms remains unclear.

The Groverian measure of entanglement for pure quantum states of multiple qubits provides an operational interpretation in terms of the success probability of certain quantum algorithms [21]. More precisely, the Groverian measure of a state $|\psi\rangle$ it is related to the success probability of Grover's search algotirhm when this state is used as the initial state. A pre-precessing stage is allowed in which an arbitrary local unitary operator is applied to each qubit. These operators are optimized in order to obtain the maximal success probability of the algorithm, $P_{\max}$. The Groverian measure is given by $G(|\psi\rangle) = \sqrt{1 - P_{\max}}$ [21]. For a state $|\psi\rangle$ of n qubits, the entanglement evaluated by this measure is, in fact, the entanglement between n parties, where each of them holds a single qubit. The Groverian measure has been used in order to characterize quantum states of high symmetry such as the Greenberg-Horne-Zeilinger (GHZ) and the W states [22]. It has also been used to evaluate the entanglement produced by quantum algorithms such as Grover's algorithm [22] and Shor's algorithm [23]. The Groverian measure was also generalized to the case of mixed states [24].

Consider a quantum state $|\psi\rangle$ of n qubits. These qubits can be partitioned into any $m \leq n$ parties, each holds one or more qubits. In this paper we present a generalized Groverian measure which quantifies the parties for any desired partition. This is done by allowing any unitary operators within each partition. This essentially changes the meaning of locality to encompass the whole party, enabling a more complete characterization of quantum states of multiple qubits.

The paper is organized as follows. In Sec. II we briefly describe Grover's search algotirhm. In Sec. III we review the Groverian entanglement measure. In Sec. IV we present the generalized Groverian measure that applies for any desired partition of the quantum state. In Sec. V we present an efficient numerical procedure for the calculation of the generalized Groverian measure. We

use this measure in Sec. VI to characterize certain pure quantum states of high symmetry. A brief discussion is presented in Sec. VII. The results are summarized in Sec. VIII.

II. GROVER'S SEARCH ALGORITHM

Grover's algorithm performs a search for a marked element m in a search space D containing N elements. We assume, for convenience, that $N = 2^n$, where n is an integer. This way, the elements of D can be represented by an n-qubit register $|x\rangle = |x_1, x_2, \ldots, x_n\rangle$, with the computational basis states $|i\rangle$, $i = 0, \ldots, N-1$. The meaning of marking the element m, is that there is a function $f : D \to \{0, 1\}$, such that $f = 1$ for the marked elements, and $f = 0$ for the rest. To solve this search problem on a classical computer one needs to evaluate f for each element, one by one, until the marked state is found. Thus, on average, $N/2$ evaluations of f are required and N in the worst case. On a quantum computer, where f can be evaluated *coherently*, a sequence of unitary operations, called Grover's algorithm and denoted by U_G, can locate a marked element using only $O(\sqrt{N})$ coherent queries of f [2, 3]. The algorithm is based on a unitary operator, called a quantum oracle, with the ability to recognize the marked states. Starting with the equal superposition state,

$$|\eta\rangle = \sum_{i=0}^{N-1} |i\rangle, \tag{1}$$

and applying the operator U_G one obtains the state

$$U_G|\eta\rangle = |m\rangle + O(1/N), \tag{2}$$

which is then measured. The success probability of the algorithm is almost unity. The adjoint equation takes the form $\langle\eta| = \langle m|U_G + O(1/N)$. If an arbitrary pure state, $|\psi\rangle$, is used as the initial state instead of the state $|\eta\rangle$, the success probability is reduced to

$$P_s = |\langle m|U_G|\psi\rangle|^2 + O(1/N). \tag{3}$$

Using Eq. (2) we obtain

$$P_s = |\langle\eta|\psi\rangle|^2 + O(1/N), \tag{4}$$

namely, the success probability is determined by the overlap between $|\psi\rangle$ and the equal superposition $|\eta\rangle$ [21, 25].

III. THE GROVERIAN ENTANGLEMENT MEASURE

Consider Grover's search algorithm, in which an arbitrary pure state $|\psi\rangle$ is used as the initial state. Before applying the operator U_G, there is a pre-processing stage in which arbitrary local unitary operators U_1, U_2, ..., U_n are applied on the n qubits in the register (Fig. 1). These operators are chosen such that the success probability of the algorithm would be maximized. The maximal success probability is thus given by

$$P_{\max} = \max_{U_1, U_2, \ldots, U_n} |\langle m|U_G(U_1 \otimes \ldots \otimes U_n)|\psi\rangle|^2. \tag{5}$$

Using Eq. (2), this can be re-written as

$$P_{\max} = \max_{U_1, U_2, \ldots, U_n} |\langle\eta|U_1 \otimes \ldots \otimes U_n|\psi\rangle|^2, \tag{6}$$

or

$$P_{\max} = \max_{|\phi\rangle \in T} |\langle\phi|\psi\rangle|^2, \tag{7}$$

where T is the space of all tensor product states of the form

$$|\phi\rangle = |\phi_1\rangle \otimes \ldots \otimes |\phi_n\rangle. \tag{8}$$

The Groverian measure is given by

$$G(\psi) = \sqrt{1 - P_{\max}}, \tag{9}$$

For the case of pure states, for which $G(\psi)$ is defined, it is closely related to an entanglement measure introduced in Refs. [14, 15, 18] for both pure and mixed states and was shown to be an entanglement monotone. This measure can be interpreted as the distance between the given state and the nearest separable state. It is expressed in terms of the fidelity between the two states. Based on these results, it was shown [21] that $G(\psi)$ satisfies (a) $G(\psi) \geq 0$, with equality only when $|\psi\rangle$ is a product state; (b) $G(\psi)$ cannot be increased using local operations and classical communication (LOCC). Therefore, $G(\psi)$ is an entanglement monotone for pure states. A related result was obtained in Ref. [26], where it was shown that the evolution of the quantum state during the iteration of Grover's algorithm corresponds to the shortest path in Hilbert space using a suitable metric.

IV. THE GENERALIZED GROVERIAN MEASURE

Consider a quantum state $|\psi\rangle$ of n qubits. In the original Groverian measure each qubit belongs to a separate party. The measure quantifies the entanglement between all these parties. This is a natural partitioning scheme for states created by quantum algorithms. The resulting measure can be considered as an intrinsic property of the state itself. However, consider a situation in which

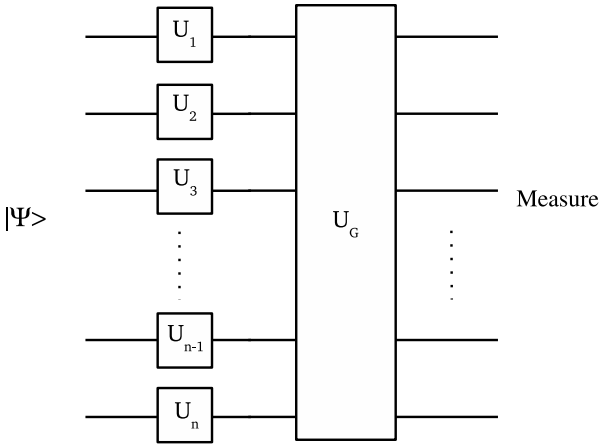

FIG. 1: The quantum circuit that exemplifies the operational meaning of the Groverian entanglement measure $G(\psi)$. A pure state $|\psi\rangle$ of n qubits is inserted as the input state. In the pre-processing stage, a local unitary operator is applied to each qubit before the resulting state is fed into Grover's algorithm. The local unitary operators U_i, $i = 1, \ldots, n$ are optimized in order to maximize the success probability of the search algorithm for the given initial state $|\psi\rangle$.

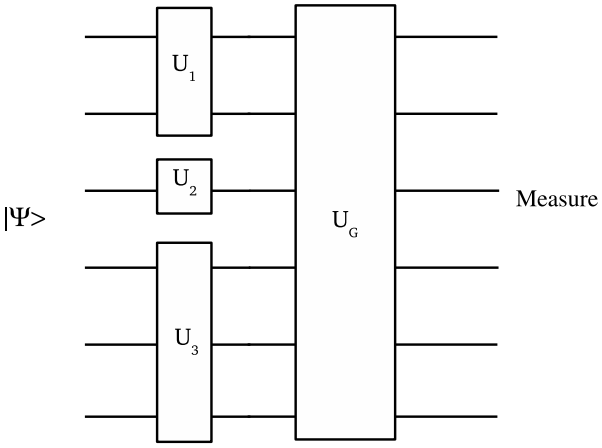

FIG. 2: The quantum circuit that exemplifies the operational meaning of the generalized Groverian measure $G(\psi)$, for n qubits divided in a certain way between m parties. In this example, a pure state $|\psi\rangle$ of six qubits, which is divided between three parties, is inserted as the input state into Grover's algorithm. In the pre-processing stage, a local unitary operator is applied on the qubits held by each party before the resulting state is fed into Grover's algorithm. The local unitary operators U_1, U_2, U_3 are optimized in order to maximize the success probability of the search algorithm, for the given initial state $|\psi\rangle$.

$m \leq n$ different parties share the quantum state, where each party holds one or more qubits. These parties wish to cooperate and perform Grover's search algorithm on the whole state. In this situation, in order to maximize the success probability, the operators U_i, $i = 1, \ldots, m$, should no longer be limited to single qubits. Instead, the operator U_i applies on all the qubits in partition i. This enables to quantify the inter-party entanglement, removing the intra-party entanglement. The quantum circuit that demonstrates the evaluation of the generalized Groverian measure for the state $|\psi\rangle$ with any desired partition is shown in Fig. 2. The generalized Groverian measure is given by Eq. (9) where Eq. (8) is replaced by $|\phi\rangle = |\phi_1\rangle \otimes \ldots \otimes |\phi_m\rangle$, where $|\phi_i\rangle$ is a state of partition i. Clearly, the generalized Groverian measure is an entanglement monotone.

V. NUMERICAL EVALUATION OF THE GENERALIZED GROVERIAN MEASURE

For a given partition of m parties, the generalized Groverian measure is expressed in terms of the maximal success probability

$$P_{\max} = \max_{|\phi_i\rangle} |\langle\phi_1| \otimes \ldots \otimes \langle\phi_m|\psi\rangle|^2, \tag{10}$$

where the maximization is over all possible states $|\phi_i\rangle$ of each partition, i. This calls for a convenient parametrization of the state of each partition. Consider a partition i that includes one qubit. The state of this partition can be expressed by

$$|\phi_i\rangle = e^{i\gamma_0} \cos\theta_0 |0\rangle + e^{i\gamma_1} \sin\theta_0 |1\rangle. \tag{11}$$

In case that the partition includes two qubits, its state can be expressed by

$$\begin{aligned} |\phi_i\rangle &= e^{i\gamma_0} \cos\theta_0 |0\rangle + \sin\theta_0 [e^{i\gamma_1} \cos\theta_1 |1\rangle \\ &+ \sin\theta_1 (e^{i\gamma_2} \cos\theta_2 |2\rangle + e^{i\gamma_3} \sin\theta_2 |3\rangle)]. \end{aligned} \tag{12}$$

This parametrization can be generalized to any number of qubits in partition i. Using this parametrization, one can express the overlap function

$$f = \langle\phi_1| \otimes \ldots \otimes \langle\phi_m|\psi\rangle, \tag{13}$$

in terms of the θ_k's and γ_k's of all the partitions. In fact, f is simply a sum of products of sine, cosine and exponential functions of the θ_k's and γ_k's. At this point, the steepest descent algorithm can be applied to maximize $|f|$. However, a more efficient maximization procedure can be obtained as follows.

For a given partition, one can express f as a function of θ_0 and γ_0, fixing all the other parameters θ_k and γ_k at this and all other partitions, in the form

$$f = c_0 \sin\theta_0 + e^{i\gamma_0} d_0 \cos\theta_0. \tag{14}$$

The values of $c_0 = |c_0| e^{i\alpha_0}$ and $d_0 = |d_0| e^{i\beta_0}$ depend on all the fixed parameters. The maximization of $|f|^2$ vs. θ_0 and γ_0 leads to

$$|f|^2 \rightarrow |c_0|^2 + |d_0|^2. \quad (15)$$

The values of γ_0 and θ_0 at which this maximization is obtained are

$$\begin{aligned} \gamma_0 &\rightarrow \beta_0 - \alpha_0 \\ \cos\theta_0 &\rightarrow \frac{|c_0|}{\sqrt{|c_0|^2 + |d_0|^2}}, \end{aligned} \quad (16)$$

where the sign of θ_0 is the same as the sign of $|d_0| - |c_0|$.

Note that the ordering of the states within each partition is arbitrary. Therefore, in order to perform the same procedure for θ_1 and γ_1, the parametrization of the two-qubit partition in Eq. (12) can be changed to

$$\begin{aligned} |\phi_i\rangle &= e^{i\gamma_1}\cos\theta_1|1\rangle + \sin\theta_1[e^{i\gamma_2}\cos\theta_2|2\rangle \\ &+ \sin\theta_2(e^{i\gamma_3}\cos\theta_3|3\rangle + e^{i\gamma_0}\sin\theta_3|0\rangle)]. \end{aligned} \quad (17)$$

In practice, the optimization procedure consists of iterations of the following steps: (a) Randomly choose a basis state $|p\rangle$ in one of the m partitions; (b) Reparamietrize the state of the chosen partition such that $|p\rangle$ will be the left-most state in Eq. (12); (c) Reset θ_p and γ_p in the chosen partition according to Eq. (16) to maximize $|f|^2$, while fixing all the other parameters.

VI. RESULTS

Using the numerical tools described above, it is possible to evaluate the generalized Groverian entanglement of any pure quantum state for any given partition. Here we demonstrate this approach for pure quantum states of high symmetry, namely the generalized GHZ state and the W state.

Consider the generalized GHZ state of three qubits

$$|\psi\rangle = a_0|000\rangle + a_1|111\rangle. \quad (18)$$

The three-party case, in which each party holds one qubit was considered before [22]. It was found that

$$P_{\max} = \max(|a_0|^2, |a_1|^2). \quad (19)$$

We will now evaluate the generalized Groverian measure for the case in which one party holds two qubits and the second party holds a single qubit. A general pure state of the first party can be expressed by

$$\begin{aligned} |\phi_1\rangle &= e^{i\gamma_0}\cos\theta_0|00\rangle + e^{i\gamma_1}\sin\theta_0\cos\theta_1|01\rangle \\ &+ e^{i\gamma_2}\sin\theta_0\sin\theta_1\cos\theta_2|10\rangle \\ &+ e^{i\gamma_3}\sin\theta_0\sin\theta_1\sin\theta_2|11\rangle, \end{aligned} \quad (20)$$

while a general pure state of the second party is given by

$$|\phi_2\rangle = e^{i\gamma_4}\cos\theta_4|0\rangle + e^{i\gamma_5}\sin\theta_4|1\rangle. \quad (21)$$

The overlap function will take the form

$$\begin{aligned} f &= e^{i\gamma_0}e^{i\gamma_4}\cos\theta_0\cos\theta_4 a_0 \\ &+ e^{i\gamma_3}e^{i\gamma_5}\sin\theta_0\sin\theta_1\sin\theta_2\sin\theta_4 a_1. \end{aligned} \quad (22)$$

The maximization of $|f|^2$ vs. all the θ_i's and γ_i's will lead to Eq. (19). This means that for the generalized GHZ state, the generalized Groverian measure does not depend on the partition. It can be shown that this result applies to generalized GHZ states with any number of qubits and any partition. This can be interpreted as if generalized GHZ states carry only bipartite entanglement, in agreement with previous studies [27].

Another family of highly symmetric pure states of multiple qubits is the class of W states. The W state of n qubits is given by

$$|\psi\rangle = \frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}|2^i\rangle, \quad (23)$$

namely it is the equal superposition of all basis states in which one qubit is 1 and all the rest are 0. This class of states was found to have $P_{\max} = (1-1/n)^{n-1}$ [22].

We will now extend this analysis to more general partitions of the n-qubit W state. First, we consider the bipartite case. In this case, the generalized Groverian entanglement is equal to the maximal eigenvalue of the reduced density matrix, traced over one of the two parties [21]. Consider the simple case in which one party includes a single qubit, while the other party includes all the other qubits. In this case we find that $P_{\max} = (n-1)/n$. In the general two-party case, one party includes k qubits and the other includes $n-k$ qubits. In this case we find that $P_{\max} = \max(k/n, 1-k/n)$.

For more that two parties, the analogy between the generalized Groverian measure and the largest eigenvalue of the reduced density matrix does not apply. Thus, the evaluation of the generalized Groverian measure can be performed analytically for a few simple cases, and in general requires the computational procedure described above.

Consider the n-qubit W state. Here we focus on a simple set of partitions to m parties, in which $m-1$ parties include one qubit each, and the last party includes all the remaining qubits. In Table I we present $P_{\max}$ for W states of $n = 2, \ldots, 7$ qubits divided between $m = 1, \ldots, n$ parties. The results in the first two rows as well as the main diagonal were obtained analytically as well as by the numerical procedure. The rest of the results were obtained numerically. Those results that appear as exact integer fraction were identified as such based on the numerical results. In four other cases, we could not identify such exact fractions.

TABLE I: The success probability $P_{\max}$ obtained for states of the W class. Each column corresponds to the W states with a given number of qubits, $n = 1, \ldots, 7$. Each row corresponds to a given number of partitions, $m = 1, \ldots, n$. Since there can be many ways to partition n qubits into m parties, we focused on a specific class of partitions in which $m-1$ parties hold one qubit each and all the remaining qubits are in one party.

Partitions	1 bit	2 bits	3 bits	4 bits	5 bits	6 bits	7 bits
1	1	1	1	1	1	1	1
2		1/2	2/3	3/4	4/5	5/6	6/7
3			$(2/3)^2$	2/4	3/5	4/6	5/7
4				$(3/4)^3$	0.4408	3/6	4/7
5					$(4/5)^4$	0.4198	0.4494
6						$(5/6)^5$	0.4084
7							$(6/7)^6$

VII. DISCUSSION

Consider a pure quantum state $|\psi\rangle$ of n qubits. The number of ways to divide these n qubits into m parties is given by the binomial coefficient C_n^m. For each of these partitions, one can evaluate the generalized Groverian measure $G(\psi)$, that quantifies the m-partite entanglement between these parties. In this analysis, locality is defined according to the partition, so that all the operations that are performed within a single partition are considered as local. Using this approach, one can identify the partition for which $G(\psi)$ is maximal among all the partitions that include m parties, and denote its value as $G_m(\psi)$. This quantity satisfies a monotonicity relation of the form $G_m(\psi) \leq G_{m+1}(\psi)$, where $G_1(\psi) = 0$. This means that splitting of parties tends to increase this measure of multipartite entanglement while merging of parties tends to decrease it.

Furthermore, the interesting question of state ordering may be addressed using this measure. It would be interesting to find pairs of states, $|\psi_1\rangle$ and $|\psi_2\rangle$, such that $G_{m_1}(\psi_1) < G_{m_1}(\psi_2)$ but $G_{m_2}(\psi_1) < G_{m_2}(\psi_2)$ for some integers m_1 and m_2.

VIII. SUMMARY

In summary, we have presented a generalization of the Groverian entanglement measure of multiple quibits to the case in which the qubits are divided into any desired partition. The generalized measure quantifies the multipartite entanglement between these partitions. To demonstrate this measure we evaluated it for a variety of pure quantum states using a combination of analytical and numerical methods. In particular, we have studied the entanglement of highly symmetric states of multiple qubits such as the generalized GHZ states and the W states.

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